Using the initial condition $f(0) = 0$ we find $C = 1$, so $f(r) = (r+1)^m(mr-1)+1$, as desired.

Editor's note: David and John also submitted a second solution to this problem that was similar to Solution 2 above.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Eastern Connecticut State University, Willimantic, CT David E. Manes SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1} + \sqrt[4]{15b^3c+1} + \sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c) + \frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since $15a^3b+1$ can be represented as $(2a)^3 \cdot \dfrac{15b+\dfrac{1}{a^3}}{8}$ then by AM-GM Inequality we obtain

$$\sum_{cyc} \sqrt[4]{15a^3b+1} = \sum_{cyc} \sqrt[4]{(2a)^3 \cdot \frac{15b+\dfrac{1}{a^3}}{8}} \le \sum_{cyc} \frac{3\cdot(2a) + \dfrac{15b+\dfrac{1}{a^3}}{8}}{4}$$

$$= \sum_{cyc} \frac{48a+15b+\dfrac{1}{a^3}}{32}$$

$$\le \frac{63}{32}(a+b+c) + \frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that

$$\sqrt[4]{11+15x^4} \le \frac{63}{32}x + \frac{1}{32x^3},\ \ x>0. \qquad (1)$$

Indeed,

$$\left(\frac{63}{32}x+\frac{1}{32x^3}\right)^4-\left(1+15x^4\right)=\frac{(x-1)^2(x+1)^2\left(x^2+1\right)^2\left(24321x^8+254x^4+1\right)}{2^{20}x^{12}}\geq 0$$

We replace x by $\sqrt[4]{a^3b}$ in (1) and use the AM$-$GM inequality to obtain

$$\sqrt[4]{1+15a^3b}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{a^9b^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot a+\frac{1}{4}\cdot b\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{a^3}+\frac{1}{4}\cdot\frac{1}{b^3}\right). \tag{2}$$

Similarly,

$$\sqrt[4]{1+15b^3c}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{b^9c^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot b+\frac{1}{4}\cdot c\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{b^3}+\frac{1}{4}\cdot\frac{1}{c^3}\right). \tag{3}$$

$$\sqrt[4]{1+15c^3a}\leq\frac{63}{32}\sqrt[4]{cb^3a}+\frac{1}{32\sqrt[4]{c^9a^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot c+\frac{1}{4}\cdot a\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{c^3}+\frac{1}{4}\cdot\frac{1}{a^3}\right). \tag{4}$$

We complete the proof by adding (2), (3), and (4).

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Let $f:(0,\infty)\longrightarrow\Re$ and $f(x)=\sqrt[4]{x}$, which is concave on $(0,\infty)$. Therefore,

$$\sqrt[4]{x}=f(x)\leq f(t)+f'(t)(x-t)=\sqrt[4]{t}+\frac{1}{4\sqrt[4]{t^3}}(x-t),\forall x,t>0.$$

Let $x=15a^3b+1$ and $t=16a^4$. Then we have:

$$\sqrt[4]{15a^3b+1}\leq 2a+\frac{1}{32a^3}\left(15a^3b+1-16a^4\right)=2a+\frac{1}{32}\left(15b+\frac{1}{a^3}-16a\right).$$

Summing the analogous upper bounds on the other two terms, gives

$$\begin{aligned}\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} &\leq 2\sum a+\frac{15}{32}\sum a-\frac{1}{2}\sum a+\frac{1}{32}\sum\frac{1}{a^3}\\ &= \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).\end{aligned}$$

Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5360:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n\geq 1$ be an integer and let

$$I_n=\int_0^\infty\frac{\arctan x}{(1+x^2)^n}dx.$$

Prove that